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Surmise relations between tests—mathematical considerations ☆

Silke Brandt*, Dietrich Albert, Cord Hockemeyer

Department of Psychology, Karl-Franzens-Universität Graz, Universitatsplatz 2/III, A-8010 Graz, Austria

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Abstract

In 1985, Doignon and Falmagne introduced *surmise relations* for representing prerequisite relationships between items within a body of information for the assessment of knowledge. Often it is useful to partition such a body of information into sub-collections. As we are primarily interested in psychological applications, we refer to these sub-collections as *tests*.

We extend the concept of surmise relations between items *within* tests to surmise relations *between* tests. Three different kinds of surmise relations between tests are investigated with respect to their properties. Furthermore, the corresponding knowledge spaces for tests and their bases are introduced. The relationship of this set theoretical approach to a Boolean matrix representation is discussed.

Finally, we give a short overview about the further research regarding this mathematical model. It will be the foundation for a software system that will be used for analyzing test data. Other applications in fields like curriculum development and structuring hyper-texts can easily be imagined.

Keywords: Surmise relation between tests; Test knowledge space; Non-numerical test theory; Prerequisite relationship; Knowledge space theory

1. Introduction

Our work is based upon the theory of knowledge spaces, which was originally introduced by Doignon and Falmagne [5,6] and a talk by Albert [1].

☆ Expanded version of a paper presented at the Meeting on Ordinal and Symbolic Data Analysis (Amherst, MA, September 1998) and published in the Electronic Notes on Discrete Mathematics [3].

* Corresponding author.

E-mail addresses: silke.brandt@talk21.com (S. Brandt), dietrich.albert@uni-graz.at (D. Albert), cord.hockemeyer@uni-graz.at (C. Hockemeyer).

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Knowledge space theory uses prerequisite relationships between items within a body of information for the assessment and training of knowledge. First some basic definitions of Doignon and Falmagne will be presented.

Definition 1. A *knowledge structure* is a pair $(Q, \mathscr{K})$ in which $Q \neq \emptyset$, $\mathscr{K} \subseteq 2^Q$, $\emptyset \in \mathscr{K}$ and $Q \in \mathscr{K}$. The set Q is called the *domain* of the knowledge structure and its elements are called *items*. We also say that $\mathscr{K}$ is a knowledge structure on a set Q. The elements of $\mathscr{K}$ are called *knowledge states*.

In our psychological interpretation, we primarily consider Q as a set of problems or questions (e.g. a test in arithmetics). The knowledge state of a person is then the set of all problems that this person is capable of solving. The knowledge structure $\mathscr{K}$ is the collection of all occurring knowledge states.

Definition 2. A knowledge structure $(Q, \mathscr{K})$ is called a *knowledge space* iff $\mathscr{K}$ is closed under union. A knowledge space $(Q, \mathscr{K})$ is called *quasi-ordinal* iff $\mathscr{K}$ is closed under intersection.

Let $(Q, \mathscr{K})$ be a knowledge structure, $x \in Q$. Then $\mathscr{K}_x$ denotes the collection of all knowledge states containing the item x:

$$\mathscr{K}_x := \{K \in \mathscr{K} \mid x \in K\}.$$

$$\bigcap \mathscr{K}_x := \bigcap_{K \in \mathscr{K}_x} K.$$

Definition 3. Let $(Q, \mathscr{K})$ denote a knowledge structure, $x, y \in Q$. A *notion* is a set

$$x^* := \{y \in Q \mid \mathscr{K}_x = \mathscr{K}_y\}.$$

The collection Q^* of all notions is a partition of Q. When two items belong to the same notion, we say that they are *equally informative*. A knowledge structure, in which each notion contains a single item, is called *discriminative*.

A discriminative knowledge structure can always be obtained from an arbitrary knowledge structure $(Q, \mathscr{K})$ by forming the notions, on constructing the knowledge structure $\mathscr{K}^*$ induced by $\mathscr{K}$ on Q^* through the definition $\mathscr{K}^* := \{K^* \mid K \in \mathscr{K}\}$ where, for any $K \in \mathscr{K}$ we have $K^* := \{x^* \mid x \in K\}$.

In the following, we will only consider discriminative quasi-ordinal knowledge spaces.

We formalize prerequisite relationships on the set Q for a quasi-ordinal knowledge space $(Q, \mathscr{K})$.

Definition 4. For a quasi-ordinal knowledge space $(Q, \mathscr{K})$, the relation $S \subseteq Q \times Q$ defined by

$$ySx \Leftrightarrow y \in \bigcap \mathscr{K}_x \quad \forall x, \quad y \in Q$$

Fig. 1. y is a prerequisite of x.

is called the *surmise relation* of the knowledge space. When ySx holds, we say that y is *surmisable* from x.

The surmise relation of a quasi-ordinal knowledge space is a quasi-order, i.e. it is reflexive and transitive; the surmise relation of a discriminative quasi-ordinal knowledge space is a partial order, i.e. reflexive, transitive, and anti-symmetric.

ySx holds iff y is an element of all the knowledge states which contain the item x. Thus, for our interpretation, each person, who masters problem x, also masters problem y. y is a prerequisite for x. Thus, from the performance of problem x we can surmise the performance of problem y (see Fig. 1).

2. Surmise relations between tests

Till now we regarded single items and surmise relations between these items *within* a body of information. Often it is useful to partition such a body of information into special fields. As we are mostly interested in psychological applications, we refer to these special fields as *tests*, but, generalized, it is of course also possible to regard, e.g., courses in curricula instead of tests [2]. We consider a partition of the whole set of items Q into tests $A, B, C, \ldots$, where $Q = A \cup B \cup C \ldots, A, B, C, \ldots \neq \emptyset$ and pairwise disjoint. In the following let $\mathscr{T} = \{A, B, C, \ldots\}$denote the whole set of tests. We now want to investigate the relations and dependencies between these tests. Therefore, we extend the concept of surmise relations between items to surmise relations between tests [1,3]. For $x \in Q$ and $B \in \mathscr{T}$ let $B_x := B \cap \bigcap \mathscr{K}_x$.

Definition 5. The relation $\dot{\mathscr{S}} \subseteq \mathscr{T} \times \mathscr{T}$ defined by

$$B \dot{\mathscr{S}} A \Leftrightarrow \exists a \in A \colon B_a \neq \emptyset \quad \forall A, \ B \in \mathscr{T}$$

is called *surmise relation between tests*. When $B \dot{\mathscr{S}} A$ holds we say A and B are in *surmise relation from A to B* or shorter: the pair (B, A) is in surmise relation.

Surmise relations between tests are interpreted in the following way: For a given item or set of items in test A a person is able to perform, we can surmise at least the performance of a nonempty subset of test B (see Fig. 2). The ability to perform these test B items is a prerequisite for performing the test A items. The surmise relation or the complementary prerequisite relation on a set of tests may—to some extent—correspond to the sequence for acquiring the different abilities or skills during a developmental or educational process, e.g. character recognition (test B) may be a

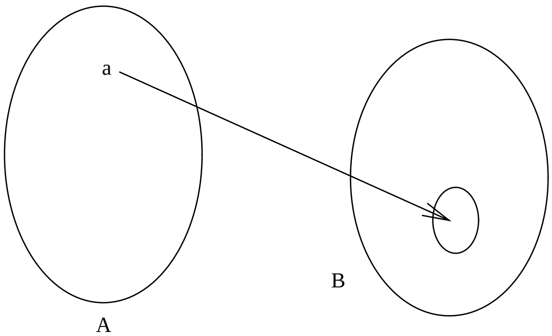

Fig. 2. A and B are in surmise relation from A to B.

prerequisite of word identification (test A). Thus, from a particular performance in test A a minimum performance in test B can be surmised. The performance in test B, however, can be higher than the necessary minimum as a development or a training in B may happen without improving the performance in A. The surmise relation between tests was introduced by Albert [1].

Now we want to investigate the properties of surmise relations between tests. The question occurs whether it is possible to transfer the properties of surmise relations between items to surmise relations between tests. As already said before, the surmise relation between items is a quasi order, that is it is reflexive and transitive.

Proposition 6. *The surmise relation between tests is reflexive, as well.*

Proof.

$$\forall a \in A\colon \exists K \in \mathscr{K} \text{ with } a \in K \text{ (as } Q \in \mathscr{K} \wedge a \in Q)$$

$$\Rightarrow \forall a \in A\colon a \in \bigcap \mathscr{K}_a$$

$$\Rightarrow \forall a \in A\colon a \in A \cap \bigcap \mathscr{K}_a = A_a \Rightarrow A \dot{\mathscr{S}} A. \qquad \square$$

See Fig. 3.

Proposition 7. *The surmise relation between tests is not necessarily transitive.*

Proof.

Suppose the surmise relation between tests is transitive.

We will show a counterexample :

Let S be a surmise relation on $Q = \{x_1, \ldots, x_4\}$ with $x_2 S x_1, x_4 S x_3$.

Consider the partition of Q into the tests $A = \{x_1\}$, $B = \{x_2, x_3\}$, and $C = \{x_4\}$.

Then for the corresponding surmise relation between tests $\dot{\mathscr{S}}$ the following holds:

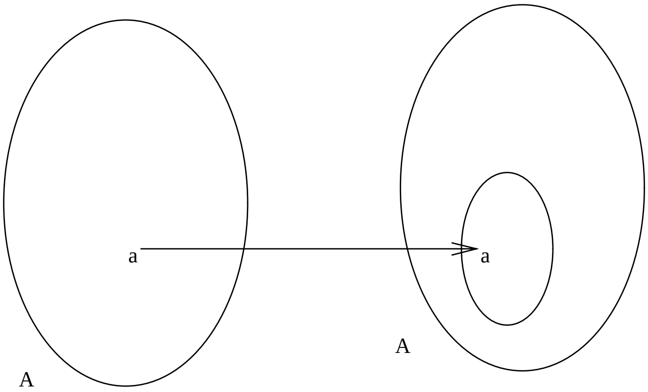

Fig. 3. $A \dot{\mathscr{S}} A$.

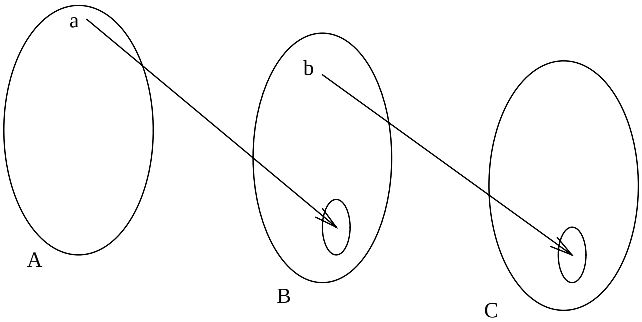

Fig. 4. $C \dot{\mathscr{S}} B$ and $B \dot{\mathscr{S}} A$, but $C \dot{\not\mathscr{S}} A$.

$C \dot{\mathscr{S}} B \wedge B \dot{\mathscr{S}} A$ but it is not the case that $C \dot{\mathscr{S}} A$.
Thus, the surmise relation between tests is not necessarily transitive
(see Fig. 4). □

Therefore, the surmise relation between tests is not a quasi order. However, there are special cases for which transitivity holds though. The first case occurs, if the surmise relation between tests is *left-covering*.

Definition 8. A and B are in *left-covering surmise relation from A to B* $\Leftrightarrow \forall a \in A$: $B_a \neq \emptyset$.
Notation: $B \dot{\mathscr{S}}_l A$.

That is, from the performance of any item in test A we can surmise the performance of a nonempty subset of items in test B (see Fig. 5).

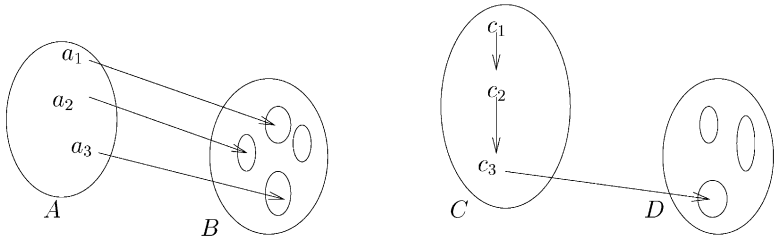

Fig. 5. $B\,\dot{\mathscr{S}}_l\,A$ and $D\,\dot{\mathscr{S}}_l\,C$.

The surmise relation between tests is called left-covering, iff $\forall A,B\in\mathscr{T}$: $B\,\dot{\mathscr{S}}\,A \Rightarrow B\,\dot{\mathscr{S}}_l\,A$ holds.

Lemma 9. $\dot{\mathscr{S}}_l \subseteq \dot{\mathscr{S}}$

Proof.

$$\begin{aligned}
&\text{Let } A,B\in\mathscr{T},\ (B,A)\in\dot{\mathscr{S}}_l\\
&\quad\Rightarrow \forall a\in A\ \exists b\in B\colon bSa\\
&\quad\Rightarrow \exists a\in A,\ \exists b\in B\colon bSa\\
&\quad\Rightarrow (B,A)\in\dot{\mathscr{S}}. \qquad \square
\end{aligned}$$

Corollary 10. *$\dot{\mathscr{S}}_l$ is reflexive on $\mathscr{T}$.*

Proof. See Lemmas 9 and 6. $\square$

Lemma 11. *$\dot{\mathscr{S}}_l$ is transitive on $\mathscr{T}$.*

Proof.

$$\begin{aligned}
&\text{Suppose } C\,\dot{\mathscr{S}}_l\,B \wedge B\,\dot{\mathscr{S}}_l\,A\\
&\quad\Rightarrow \forall b\in B\ \exists c\in C\colon c\in\bigcap\mathscr{K}_b \wedge \forall a\in A\ \exists b\in B\colon b\in\bigcap\mathscr{K}_a\\
&\quad\Rightarrow \forall a\in A\ \exists b\in B,\ c\in C\colon b\in\bigcap\mathscr{K}_a \wedge c\in\bigcap\mathscr{K}_b\\
&\quad\Rightarrow \forall a\in A\ \exists c\in C\colon c\in\bigcap\mathscr{K}_a\\
&\quad\Rightarrow C\,\dot{\mathscr{S}}_l\,A. \qquad \square
\end{aligned}$$

See Fig. 6.

The second special case occurs, if the surmise relation between tests is *right-covering*.

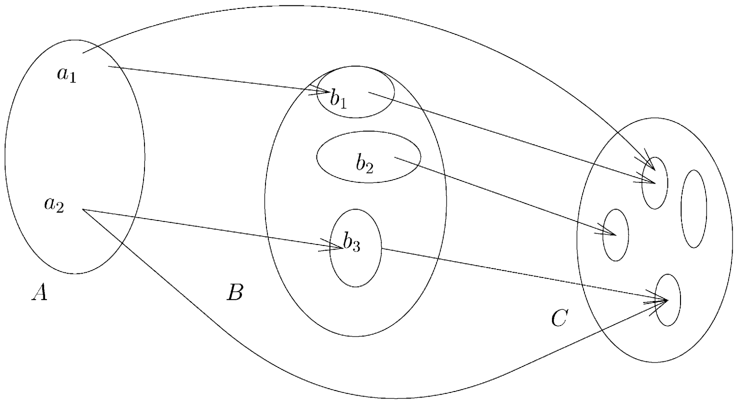

Fig. 6. $C\,\dot{\mathscr{S}}_l\,B \wedge B\,\dot{\mathscr{S}}_l\,A \Rightarrow C\,\dot{\mathscr{S}}_l\,A$.

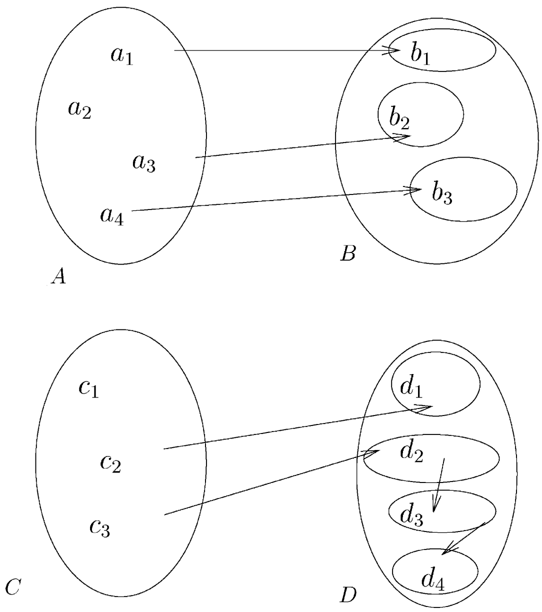

Fig. 7. $A\,\dot{\mathscr{S}}_r\,B$ and $C\,\dot{\mathscr{S}}_r\,D$.

Definition 12. A and B are in *right-covering surmise relation from A to B* $\Leftrightarrow \bigcup_{a\in A} B_a = B$.

Notation: $B\,\dot{\mathscr{S}}_r\,A$ (see Fig. 7).

For all items b in test B, there exists an item a in test A for which $b \in B_a$ and, thus, bSa holds. From the performance of the whole test A the performance of the whole test B can be surmised. The whole test B is a prerequisite for the test A.

The surmise relation between tests is called right-covering, iff $\forall A, B \in \mathscr{T}$: $B\,\dot{\mathscr{S}}\,A \Rightarrow B\,\dot{\mathscr{S}}_r\,A$ holds.

Lemma 13. $\dot{\mathscr{S}}_r \subseteq \dot{\mathscr{S}}$.

Proof.

$$
\begin{aligned}
&\text{Let } A, B \in \mathscr{T},\ (B, A) \in \dot{\mathscr{S}}_r \\
&\quad \Rightarrow \forall b \in B\ \exists a \in A\colon bSa \\
&\quad \Rightarrow \exists b \in B,\ \exists a \in A\colon bSa \\
&\quad \Rightarrow (B, A) \in \dot{\mathscr{S}}. \qquad \Box
\end{aligned}
$$

Corollary 14. *$\dot{\mathscr{S}}_r$ is reflexive on $\mathscr{T}$.*

Proof. See Lemmas 6 and 13. $\Box$

Lemma 15. *$\dot{\mathscr{S}}_r$ is transitive on $\mathscr{T}$.*

Proof.

$$
\begin{aligned}
&\text{Suppose } C\,\dot{\mathscr{S}}_r\,B \wedge B\,\dot{\mathscr{S}}_r\,A \\
&\quad \Rightarrow \forall c \in C\ \exists b \in B\colon c \in \bigcap \mathscr{K}_b \wedge \forall b \in B \exists a \in A\colon b \in \bigcap \mathscr{K}_a \\
&\quad \Rightarrow \forall c \in C\ \exists a \in A,\ b \in B\colon c \in \bigcap \mathscr{K}_b \wedge b \in \bigcap \mathscr{K}_a \\
&\quad \Rightarrow \forall c \in C\ \exists a \in A\colon c \in \bigcap \mathscr{K}_a \\
&\quad \Rightarrow \bigcup_{a \in A} \left(C \cap \bigcap \dot{\mathscr{K}}_a \right) = C \\
&\quad \Rightarrow \bigcup_{a \in A} C_a = C \Rightarrow C\,\dot{\mathscr{S}}_r\,A. \qquad \Box
\end{aligned}
$$

See Fig. 8.

Thus, both the left-covering and the right-covering surmise relation are quasi-orders.

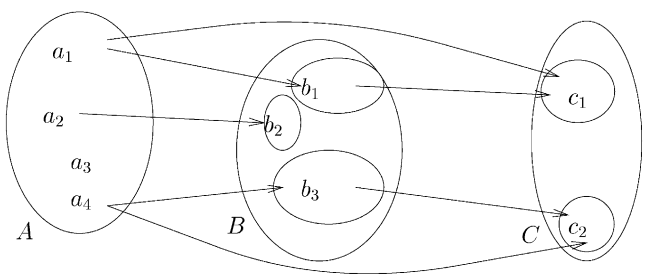

Fig. 8. $C\,\dot{\mathscr{S}}_r\,B \wedge B\,\dot{\mathscr{S}}_r\,A \Rightarrow C\,\dot{\mathscr{S}}_r\,A$.

3. Test knowledge spaces

The concept of test knowledge spaces is based upon the concept of knowledge spaces.

Definition 16. For a knowledge state $K_i \in \mathscr{K}$ and $\mathscr{T} = \{A, B, C, \ldots\}$ the n-tuple $\dot{K}_i = (A_i, B_i, \ldots)$, where $A_i = A \cap K_i$, $B_i = B \cap K_i, \ldots$ for $i \in \mathbf{N}$, is called *test knowledge state*. Let $\dot{\mathscr{K}}$ denote the collection of all test knowledge states. Then the pair $(\mathscr{T}, \dot{\mathscr{K}})$ is called *test knowledge structure*.

In our interpretation, if $\dot{K}_i$ is the test knowledge state of a person, then A_i is the subset of items in test A, which this person is capable of solving, B_i is the subset of items in test B, which this person is capable of solving, and so on.

Definition 17. A test knowledge structure $(\mathscr{T}, \dot{\mathscr{K}})$ is a *test knowledge space*, iff $\dot{\mathscr{K}}$ is closed under union. $(\mathscr{T}, \dot{\mathscr{K}})$ is a *quasi-ordinal test knowledge space* iff $\dot{\mathscr{K}}$ is closed under union and intersection.

Notice that union and intersection for n-tuples is not the same as union and intersection for sets!

Definition 18. For $\dot{K}_i = (A_i, B_i, \ldots)$ and $\dot{K}_j = (A_j, B_j, \ldots)$:

$$\dot{K}_i \dot{\cup} \dot{K}_j := (A_i \cup A_j, B_i \cup B_j, \ldots).$$
$$\dot{K}_i \dot{\cap} \dot{K}_j := (A_i \cap A_j, B_i \cap B_j, \ldots).$$

Lemma 19. *The test knowledge structure $\dot{\mathscr{K}}$ is a test knowledge space $\Leftrightarrow$ the corresponding knowledge structure $\mathscr{K}$ is a knowledge space. The test knowledge space $\dot{\mathscr{K}}$ is quasi ordinal $\Leftrightarrow$ the corresponding knowledge space $\mathscr{K}$ is quasi ordinal.*

Proof.

"$\Rightarrow$": Let $K_i, K_j \in \mathscr{K}$, $\dot{K}_i = (A_i, B_i, \ldots)$, $\dot{K}_j = (A_j, B_j, \ldots) \in \dot{\mathscr{K}}$, $\dot{\mathscr{K}}$ be closed under union

$\Rightarrow \dot{K}_i \dot{\cup} \dot{K}_j = (A_i \cup A_j,\ B_i \cup B_j, \ldots) \in \dot{\mathscr{K}}$

$\Rightarrow \exists K' \in \mathscr{K}$: $(X_i \cup X_j) = X \cap K' \quad \forall X \in \mathscr{T}$

We know : $X_i = X \cap K_i$, $X_j = X \cap K_j \quad \forall X \in \mathscr{T}$

$\Rightarrow (X \cap K_i) \cup (X \cap K_j) = X \cap K' \quad \forall X \in \mathscr{T}$

$\Rightarrow X \cap (K_i \cup K_j) = X \cap K' \quad \forall X \in \mathscr{T}$

$\Rightarrow K_i \cup K_j = K' \Rightarrow K_i \cup K_j \in \mathscr{K}$.

"$\Leftarrow$": Let $K_i, K_j \in \mathscr{K}$, $\dot{K}_i = (A_i, B_i, \ldots)$, $\dot{K}_j = (A_j, B_j, \ldots) \in \dot{\mathscr{K}}$, $\mathscr{K}$ be closed under union

$\Rightarrow K_i \cup K_j \in \mathscr{K}$

$\Rightarrow \dot{K}' := (A \cap (K_i \cup K_j), B \cap (K_i \cup K_j), \ldots) \in \dot{\mathscr{K}}$.

$$\dot{K}_i \dot{\cup} \dot{K}_j = (A_i \cup A_j,\ B_i \cup B_j, \ldots).$$
$$X_i \cup X_j = (X \cap K_i) \cup (X \cap K_j) = X \cap (K_i \cup K_j) \quad \forall X \in \mathscr{T}$$
$$\Rightarrow \dot{K}_i \cup \dot{K}_j = \dot{K}' \in \dot{\mathscr{K}}.$$

Closure under intersection : analogous. □

4. The base

Definition 20. A subcollection $\mathscr{B} \subseteq \mathscr{K}$ of states is called *base* of $\mathscr{K}$ iff the following conditions hold:

(1) All the states of $\mathscr{K}$ can be obtained by taking all arbitrary unions (including the empty union) of the states included in the subcollection $\mathscr{B}$.
$\forall K \in \mathscr{K}\ \exists K_1, \ldots, K_n \in \mathscr{B},\ n \in \mathbf{N}$, such that $K = K_1 \cup \cdots \cup K_n$.

(2) $\mathscr{B}$ is minimal in the sense that it is a subset of any other subcollection of states generating the states in $\mathscr{K}$ by taking unions of states in $\mathscr{B}$.
$\forall \mathscr{P}$ which fulfill (1), holds: $\mathscr{B} \subseteq \mathscr{P}$.

If the set Q of items is finite and the corresponding knowledge structure $\mathscr{K}$ is a knowledge space, it is always possible to find such a base for $\mathscr{K}$. In particular there exists one and only one base for each knowledge space [5,6]. Because of Corollary 19 it is easy to transfer this definition of a base for a knowledge space to the definition of a base for a test knowledge space. We only have to replace $\mathscr{K}$ by $\dot{\mathscr{K}}$, K by $\dot{K}$ and union for sets by the union defined in Definition 18.

Definition 21. $\dot{\mathscr{B}} \subseteq \dot{\mathscr{K}}$ is called *base* of $\dot{\mathscr{K}}$ iff the following conditions hold:

(1) $\forall \dot{K} \in \dot{\mathscr{K}}\ \exists \dot{K}_1, \ldots, \dot{K}_n \in \dot{\mathscr{B}}$: $\dot{K} = \dot{K}_1 \dot{\cup} \cdots \dot{\cup} \dot{K}_n$.

(2) $\forall \dot{\mathscr{P}} \subseteq \dot{\mathscr{K}}$ which fulfill (1), $\dot{\mathscr{B}} \subseteq \dot{\mathscr{P}}$ holds.

In particular, the following statement holds:

Lemma 22. *Let $(Q, \mathscr{K})$ denote a knowledge structure and $(\mathscr{T}, \dot{\mathscr{K}})$ denote the corresponding test knowledge structure. Then $\dot{\mathscr{B}} = \{(A_i, B_i, C_i, \ldots), (A_j, B_j, C_j, \ldots), \ldots\}$ is the base of $\dot{\mathscr{K}} \Leftrightarrow \mathscr{B} = \{A_i \cup B_i \cup C_i \ldots, A_j \cup B_j \cup C_j \ldots, \ldots\}$ is the base of $\mathscr{K}$.*

That is, the base $\dot{\mathscr{B}}$ of $\dot{\mathscr{K}}$ is just the set of test knowledge states corresponding to the elements of the base $\mathscr{B}$ of $\mathscr{K}$.

Proof.

Let $\mathscr{B} = \{K_1, \ldots, K_n\} \wedge \dot{\mathscr{B}} := \{\dot{K}_1, \ldots, \dot{K}_n\}$ with $\dot{K}_i := (A_i, B_i, C_i, \ldots)$,
$A_i = A \cap K_i,\ B_i = B \cap K_i,\ C_i = C \cap K_i, \ldots$ for $i \in \{1, \ldots, n\}$,
$\dot{\mathscr{B}} \subseteq \dot{\mathscr{K}} \Leftrightarrow \dot{K}_i \in \dot{\mathscr{K}}$ for $i \in \{1, \ldots, n\}$
$\Leftrightarrow K_i \in \mathscr{K}$ for $i \in \{1, \ldots, n\} \Leftrightarrow \mathscr{B} \subseteq \mathscr{K}$.
$\forall \dot{K}_m := (A_m, B_m, C_m, \ldots) \in \dot{\mathscr{K}}\ \exists \dot{K}_1, \ldots, \dot{K}_j \in \dot{\mathscr{B}}$ with $\dot{K}_m = \dot{K}_1 \dot{\cup} \cdots \dot{\cup} \dot{K}_j$

$$\Leftrightarrow \forall \dot{K}_m \in \dot{\mathscr{K}}\ \exists \dot{K}_1,\ldots,\dot{K}_j \in \dot{\mathscr{B}} \quad \text{with } A_m = \bigcup_{i=1}^{j} A_i,$$

$$B_m = \bigcup_{i=1}^{j} B_i,\ C_m = \bigcup_{i=1}^{j} C_i,\ldots$$

$$\Leftrightarrow \forall K_m \in \mathscr{K}\ \exists K_1,\ldots,K_j \in \mathscr{B} \quad \text{with } A_i = A \cap K_i,$$

$$B_i = B \cap K_i,\ C_i = C \cap K_i \quad \text{for } i = \{1,\ldots,j\}$$

$$\wedge \bigcup_{i=1}^{j} A_i = A \cap K_m,\ \bigcup_{i=1}^{j} B_i = B \cap K_m,\ldots$$

$$\Leftrightarrow \forall K_m \in \mathscr{K}\ \exists K_1,\ldots,K_j \in \mathscr{B}\colon \bigcup_{i=1}^{j} (A \cap K_i) = A \cap K_m,$$

$$\bigcup_{i=1}^{j} (B \cap K_i) = B \cap K_m,\ldots$$

$$\Leftrightarrow \forall K_m \in \mathscr{K}\ \exists K_1,\ldots,K_j \in \mathscr{B}\colon A \cap \bigcup_{i=1}^{j} K_i = A \cap K_m,\ B \cap \bigcup_{i=1}^{j} K_i = B \cap K_m,\ldots$$

$$\Leftrightarrow \forall K_m \in \mathscr{K}\ \exists K_1,\ldots,K_j \in \mathscr{B}\colon \bigcup_{i=1}^{j} K_i = K_m.$$

$\forall \dot{\mathscr{P}} \subseteq \dot{\mathscr{K}}$ which fulfill Definition 21(1), $\dot{\mathscr{B}} \subseteq \dot{\mathscr{P}}$ holds

$\Leftrightarrow \forall \mathscr{P} \subseteq \mathscr{K}$ which fulfill Definition 20(1), $\mathscr{B} \subseteq \mathscr{P}$ holds. □

Therefore, there exists exactly one base for each test knowledge space, if Q is finite. Chubb [4] gives an algorithm for constructing the base in the finite case. The base is the most compressed form for storing the list of test knowledge states. By means of the base $\dot{\mathscr{B}}$ we can infer the test knowledge space $\dot{\mathscr{K}}$, the corresponding knowledge space $\mathscr{K}$ and the surmise relation between items; moreover—and this is an important conclusion of our concept—we can also infer the surmise relation between tests and its properties as there are antisymmetry, transitivity, left- and right-coveringness by means of the base. Propositions 23–25 make it very easy to investigate the properties of the surmise relation between tests for quasi ordinal test knowledge spaces. In the following, let $\dot{\mathscr{B}} = \{\dot{K}_1,\ldots,\dot{K}_n\}$ for $i \in \{1,\ldots,n\}$ denote the base of the quasi ordinal test knowledge space $\dot{\mathscr{K}}$.

By means of $\dot{\mathscr{B}}$ we can infer the corresponding test surmise relation $\dot{\mathscr{S}}$ using Proposition 23:

Proposition 23. *$A \dot{\mathscr{S}} B \Leftrightarrow \forall \dot{K}_i \in \dot{\mathscr{B}}$ with $A_i = \emptyset$: $\bigcup B_i \subset B$.*

This proposition derives from the fact that whenever $A \dot{\mathscr{S}} B$ holds, nobody who fails to solve any item of test A will be able to solve the whole test B.

Proof.

"$\Rightarrow$": Let $A\,\dot{\mathscr{S}}\,B \Rightarrow \exists b\in B,\ \exists a\in A\colon a\in\bigcap\mathscr{K}_b$

$\Rightarrow \exists b\in B, a\in A\colon \forall K_i\in\mathscr{K}\colon (b\in K_i \Rightarrow a\in K_i)$

$\Rightarrow \exists b\in B, a\in A\colon \forall \dot{K}_i\in\dot{\mathscr{K}}\colon (b\in B_i = B\cap K_i \Rightarrow a\in A_i = A\cap K_i)$

$\Rightarrow \exists b\in B, a\in A\colon \forall \dot{K}_i\in\dot{\mathscr{K}}\colon (a\notin A_i \Rightarrow b\notin B_i)$

$\Rightarrow \forall \dot{K}_i\in\dot{\mathscr{K}}\colon (A_i=\emptyset \Rightarrow B_i\neq B)$

$\Rightarrow \forall \dot{K}_i\in\dot{\mathscr{B}}$ with $A_i=\emptyset$: $\bigcup B_i\subset B$.

"$\Leftarrow$": $\forall \dot{K}_i\in\dot{\mathscr{B}}$ with $A_i=\emptyset$: $\bigcup B_i\subset B$

$\Rightarrow \forall \dot{K}_i\in\dot{\mathscr{K}}$ with $A_i=\emptyset$: $B_i\neq B$

$\Rightarrow \forall \dot{K}_i\in\dot{\mathscr{K}}$ with $A_i=\emptyset$: $\exists b\in B$ with $b\notin B_i$.

Supposition 1: $\forall b\in B\ \exists \dot{K}_i\in\dot{\mathscr{K}}$ with $A_i=\emptyset\wedge b\in B_i$

$$\Rightarrow \text{For } \dot{K}_k := \dot{\bigcup_{A_i=\emptyset}} \dot{K}_i \text{ we have } A_k=\emptyset,\ B_k = B\wedge \dot{K}_k\in\dot{\mathscr{K}},$$

as $\dot{\mathscr{K}}$ is closed under union.

This is a contradiction to our assumption

$\Rightarrow$ Supposition 1 is wrong

$\Rightarrow \exists b\in B\colon \forall \dot{K}_i\in\dot{\mathscr{K}}$ with $A_i=\emptyset$: $b\notin B_i$

$\Rightarrow \exists b\in B\colon \forall \dot{K}_i\in\dot{\mathscr{K}}$ with $b\in B_i$: $A_i\neq\emptyset$

$\Rightarrow \exists b\in B\colon \forall \dot{K}_i\in\dot{\mathscr{K}}$ with $b\in B_i$ $\exists a\in A\colon a\in A_i$. $\quad(*)$

Supposition 2: $\forall a\in A\ \exists \dot{K}_i\colon b\in B_i\wedge a\notin A_i$

$$\Rightarrow \text{For } \dot{K}_t := \dot{\bigcap_{b\in B_i}} \dot{K}_i \text{ we have } A_t=\emptyset,\ b\in B_t\wedge K_t\in\dot{\mathscr{K}}$$

as $\dot{\mathscr{K}}$ is closed under intersection.

This is a contradiction to $(*)$

$\Rightarrow$ Supposition 2 is wrong

$\Rightarrow \exists b\in B, a\in A\colon \forall \dot{K}_i\in\dot{\mathscr{K}}\colon (b\in B_i\Rightarrow a\in A_i)$

$\Rightarrow A\,\dot{\mathscr{S}}\,B$. $\quad\square$

By means of $\dot{\mathscr{B}}$ we can also investigate whether two tests are in left-covering surmise relation:

Proposition 24. $A\,\dot{\mathscr{S}}_l\,B \Leftrightarrow \forall \dot{K}_i\in\dot{\mathscr{B}}$ *with* $B_i\neq\emptyset$: $A_i\neq\emptyset$.

Proof.

"$\Rightarrow$": Let $A\,\dot{\mathscr{S}}_l\,B \Rightarrow \forall b\in B\colon A_b\neq\emptyset$

$\Rightarrow \forall b\in B\ \exists a\in A\colon a\in\bigcap\mathscr{K}_b$

$\Rightarrow \forall K_i\in\mathscr{K}\colon (\exists b\in B\cap K_i \Rightarrow \exists a\in A\cap K_i)$

$$\Rightarrow \forall K_i \in \mathscr{K}\colon (B_i \neq \emptyset \Rightarrow A_i \neq \emptyset)$$
$$\Rightarrow \forall \dot{K}_i \in \dot{\mathscr{K}}\colon (B_i \neq \emptyset \Rightarrow A_i \neq \emptyset)$$
$$\Rightarrow \forall \dot{K}_i \in \dot{\mathscr{B}}\colon (B_i \neq \emptyset \Rightarrow A_i \neq \emptyset).$$

" $\Leftarrow$ ": $\forall \dot{K}_i \in \dot{\mathscr{B}}\colon (B_i \neq \emptyset \Rightarrow A_i \neq \emptyset)$.

Supposition: $\exists b \in B\colon A_b = \emptyset$

$$\Rightarrow \exists b \in B\colon \forall a \in A\colon a \notin \bigcap \mathscr{K}_b$$
$$\Rightarrow \exists b \in B\colon \forall a \in A \ \exists K_i \in \mathscr{K}\colon b \in K_i \wedge a \notin K_i$$
$$\Rightarrow \exists b \in B\colon \forall a \in A \ \exists \dot{K}_i \in \dot{\mathscr{K}}\colon b \in B_i = B \cap K_i \wedge a \notin A_i = a \cap K_i$$
$$\Rightarrow \exists b \in B\colon \quad \text{for } \dot{K}_j := \dot{\bigcap_{b \in B_i}} \text{ we have } A_i = \emptyset \wedge B_i \neq \emptyset \text{ (as } b \in B_i\text{)}.$$

This is a contradiction to our assumption

$\Rightarrow$ The Supposition is wrong

$\Rightarrow \forall b \in B\colon A_b \neq \emptyset \Rightarrow A\, \dot{\mathscr{S}}_l\, B.$ □

On applying Proposition 24 to any two tests in $\mathscr{T}$ we can check whether the test surmise relation $\dot{\mathscr{S}}$ on $\mathscr{T}$ is left-covering.

By means of $\dot{\mathscr{B}}$ we can also investigate Right-coveringness:

Proposition 25. $A\, \dot{\mathscr{S}}_r\, B \Leftrightarrow \forall \dot{K}_1, \ldots, \dot{K}_n \in \dot{\mathscr{B}}$ *with* $\bigcup_{i=1}^n B_i = B\colon \bigcup_{i=1}^n A_i = A$

Proof.

" $\Rightarrow$ ": Let $A\, \dot{\mathscr{S}}_r\, B \Rightarrow \bigcup_{b \in B} A_b = A$

$$\Rightarrow \bigcup_{b \in B} \left(A \cap \bigcap \mathscr{K}_b \right) = A$$
$$\Rightarrow A \cap \left(\bigcup_{b \in B} \left(\bigcap \mathscr{K}_b \right) \right) = A$$
$$\Rightarrow A \subseteq \bigcup_{b \in B} \left(\bigcap \mathscr{K}_b \right)$$
$$\Rightarrow \forall a \in A \ \exists b \in B\colon a \in \bigcap \mathscr{K}_b.$$

Suppose $\dot{K}_1, \ldots, \dot{K}_n \in \dot{\mathscr{B}}$ with $\bigcup_{j=1}^n B_j = B$

$$\Rightarrow \text{For } \dot{K}_i := \dot{K}_1 \dot{\cup} \cdots \dot{\cup} K_j \in \dot{\mathscr{K}} \bigcup_{j=1}^n B_j = B$$
$$\Rightarrow \forall b \in B\colon b \in B_i$$
$$\Rightarrow \forall a \in A \ \exists b \in B\colon b \in K_i \wedge a \in \bigcap \mathscr{K}_b$$
$$\Rightarrow \forall a \in A a \in K_i \Rightarrow A_i = A.$$

$$\text{“}\Leftarrow\text{”: } \forall \dot{K}_1,\dots,\dot{K}_n \in \dot{\mathscr{B}}\text{: } \left(\bigcup_{j=1}^{n} B_j = B \Rightarrow \bigcup_{j=1}^{n} A_j = A\right)$$

$$\Rightarrow \forall \dot{K}_i \in \dot{\mathscr{K}}\text{: } (B_i = B \Rightarrow A_i = A) \quad (*).$$

$$\Rightarrow Supposition\text{: } \exists a \in A\text{: } \forall b \in B\text{: } a \notin \bigcap \mathscr{K}_b$$

$$\Rightarrow \exists a \in A \quad \forall b \in B \; \exists K_i \in \mathscr{K}\text{: } b \in K_i \wedge a \notin K_i$$

$$\Rightarrow \exists a \in A \quad \forall b \in B \; \exists \dot{K}_i \in \dot{\mathscr{K}}\text{: } b \in B_i \wedge a \notin A_i.$$

For $\dot{K}_k := \dot{\bigcup_{a \notin A_i}}$ we have $B_k = B,\ A_k \neq A$ (as $a \notin A_k$)

$\wedge \dot{K}_k \in \dot{\mathscr{K}}$, as $\dot{\mathscr{K}}$ is closed under union.

This is a contradiction to $(*)$

$\Rightarrow$ The supposition is wrong

$$\Rightarrow \forall a \in A \; \exists b \in B\text{: } a \in \bigcap \mathscr{K}_b$$

$$\Rightarrow \forall a \in A\text{: } a \in \left(\bigcup_{b \in B} \left(\bigcap \mathscr{K}_b\right)\right)$$

$$\Rightarrow A \cap \left(\bigcup_{b \in B} \left(\bigcap \mathscr{K}_b\right)\right) = A$$

$$\Rightarrow \bigcup_{b \in B} A_b = A \Rightarrow A \, \dot{\mathscr{S}}_r \, B. \qquad \square$$

The base plays a central role as an efficient way of storing information. Test knowledge spaces are often big and thus, difficult, if not impossible to handle. For such a big test knowledge space it is essential to find a base which stores all the information about the test knowledge space and from which the corresponding test surmise relation $\dot{\mathscr{S}}$ and its properties can be inferred.

5. Relationship to Boolean matrix representations

Any binary relation R on a set can be represented by a Boolean Matrix M: label objects $x_1,\dots,x_n$ and let $M_{ij} = 1$ if $(x_i, x_j) \in R$ and $M_{ij} = 0$ if $(x_i, x_j) \notin R$.

In the following, we consider such a Boolean matrix representation for Surmise relations. Let $|Q| = n$ and S be a surmise relation on Q. Then S can be represented by the n-square Boolean matrix M with $M_{ij} = 1$ if iSj, and $M_{ij} = 0$, otherwise. Using this representation we can apply some of the results of Kim and Roush [7] regarding Group relationships and Homomorphisms of Boolean Matrix Semi groups.

Every Boolean matrix A, which represents a binary relation R, can be associated with a smaller matrix in the following way: Take a partition of the set of individuals $\{x_1,\dots x_n\}$ and divide the matrix into *blocks*. Now form the *image matrix* by replacing each zero block by a single zero and each nonzero block by a single one.

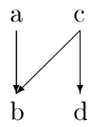

Fig. 9. Surmise relation of Example 5.1.

Example 5.1. Regard the surmise relation S on the set $Q=\{a,\ldots,d\}$, let bSa, bSc and dSc (see Fig. 9).

S can be represented by the matrix

$$M=\begin{pmatrix}1&0&0&0\\1&1&1&0\\0&0&1&0\\0&0&1&1\end{pmatrix}.$$

Consider the partition $A=\{a,b\}$, $B=\{c,d\}$, $A\cup B=Q$. The above matrix M is divided into for blocks P_{AA},P_{AB},P_{BA}, and P_{BB}:

$$M=\left(\begin{array}{cc|cc}1&0&0&0\\1&1&1&0\\\hline 0&0&1&0\\0&0&1&1\end{array}\right).$$

For the image matrix $\dot{M}$ we have $\dot{M}_{AA}=\dot{M}_{AB}=\dot{M}_{BB}=1$, as P_{AA}, P_{AB}, and P_{BB} are nonzero blocks, and $\dot{M}_{BA}=0$, as P_{BA} is a zero block.

$$\dot{M}=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

In general, each pair A,B of tests is associated with a sub matrix P_{AB} of M: the rows and columns of P_{AB} index the items in A and B, respectively (and $\dot{M}_{AB}=1$ iff there is at least one '1' in P_{AB}).

Lemma 26. *Let S be a surmise relation on the set Q and let M denote the Boolean matrix representing S. Consider a partition $A\cup B\cup C\cup\cdots=Q$. Then the surmise relation $\dot{\mathscr{S}}$ on the set $\mathscr{T}=\{A,B,C,\ldots\}$ of tests can be represented by the image matrix $\dot{M}$ of M, i.e.: $A\,\dot{\mathscr{S}}\,B\Leftrightarrow\dot{M}_{AB}=1$ for all $A,B\in\mathscr{T}$.*

Proof.

$$\begin{aligned}&\textit{Let } A,B\in\mathscr{T}.\\&\dot{M}_{AB}=1\Leftrightarrow P_{AB}\textit{ is a nonzero block of } M\\&\quad\Leftrightarrow\exists x_i\in A,\ \exists x_j\in B\colon M_{ij}=1\Leftrightarrow\exists x_i\in A,\ \exists x_j\in B\colon aSb\\&\quad\Leftrightarrow\exists x_i\in A,\ \exists x_j\in B\colon x_i\in\bigcap\mathscr{K}_{x_j}\Leftrightarrow\exists x_j\in B\colon A_{x_j}\neq\emptyset\Leftrightarrow A\,\dot{\mathscr{S}}\,B.\qquad\square\end{aligned}$$

Between tests is defined so that $\dot{M}_{AB} = 1 \Leftrightarrow M_{ij}$ for some $i \in A$ and some $j \in B$. Further, each pair A, B of tests is associated with a sub matrix P_{AB} of M: the rows and columns of P_{AB} index the items in A and B, respectively (and $\dot{M}_{AB} = 1 \Leftrightarrow$ there is at least one ‘1’ in P_{AB}). The left- (right-) covering condition is then a requirement that $\dot{M}_{AB} = 1 \Leftrightarrow P_{AB}$ has a 1 in every row (column), for all pairs A, B of tests.

Lemma 27. *The surmise relation $\dot{\mathscr{S}}$ on $\mathscr{T}$ is left-covering iff* ($\dot{M}_{AB} = 1 \Leftrightarrow P_{AB}$ *has a* 1 *in every column for all pairs* A, B *of tests*).

Proof.

$$\begin{aligned}&\text{Suppose } \dot{M}_{AB} = 1 \text{ for some } A, B \in \mathscr{T}.\\ &P_{AB} \text{ has a 1 in every column} \Leftrightarrow \forall b \in B\ \exists a \in A\colon M_{ab} = 1\\ &\quad \Leftrightarrow \forall b \in B\ \exists a \in A\colon a \in \bigcap \mathscr{K}_b \Leftrightarrow \forall b \in B\ \exists a \in A\colon A_b \neq \emptyset \Leftrightarrow A\, \dot{\mathscr{S}}_l\, B. \qquad \square\end{aligned}$$

Lemma 28. *The surmise relation $\dot{\mathscr{S}}$ on $\mathscr{T}$ is right-covering iff* ($\dot{M}_{AB} = 1 \Leftrightarrow P_{AB}$ *has a* 1 *in every row for all pairs* A, B *of tests*).

Proof.

$$\begin{aligned}&\text{Suppose } \dot{M}_{AB} = 1 \text{ for some } A, B \in \mathscr{T}.\\ &P_{AB} \text{ has a 1 in every row} \Leftrightarrow \forall a \in A\ \exists b \in B\colon M_{ab} = 1\\ &\quad \Leftrightarrow \forall a \in A\ \exists b \in B\colon aSb \Leftrightarrow \forall a \in A\ \exists b \in B\colon a \in \bigcap \mathscr{K}_b\\ &\quad \Leftrightarrow \forall a \in A\colon a \in \bigcup_{b \in B} \left(\bigcap \mathscr{K}_b\right) \Leftrightarrow A \subseteq \bigcup_{b \in B} \left(\bigcap \mathscr{K}_b\right)\\ &\quad \Leftrightarrow A \cap \left[\bigcup_{b \in B} \left(\bigcap \mathscr{K}_b\right)\right] = A \Leftrightarrow \bigcup_{b \in B} \left(A \cap \bigcap \mathscr{K}_b\right) = A\\ &\quad \Leftrightarrow \bigcup_{b \in B} A_b = A \Leftrightarrow A\, \dot{\mathscr{S}}_r\, B. \qquad \square\end{aligned}$$

Some results described by Kim and Roush [7] identify a wide set of conditions under which the surmise relation between tests is a quasi order, and include the cases of left-covering and right-covering surmise relations introduced here as special cases.

Definition 29. Let $i \in \mathbf{N}$. The surmise relation between tests $\dot{\mathscr{S}}$ on the set $\mathscr{T}$ of tests satisfies the condition G_i iff, for all $A, B \in \mathscr{T}$ with $A\, \dot{\mathscr{S}}\, B$, the following holds:

Let $X \subseteq A, |X| = i$ (or, if $i > |A|$, let $X = A$). Then $|\{b \in B \mid \exists a \in X\colon M_{ab} = 1\}| \geqslant min(i, |B|)$.

Proposition 30. *The surmise relation between tests $\dot{\mathscr{S}}$ on the set $\mathscr{T}$ of tests is transitive whenever G_i is satisfied for any i.*

Remark. Proposition 30 follows from Kim and Roush's results, who showed that whenever G_i is satisfied taking the image matrix is a multiplicative homomorphism. Transitivity of surmise relations between tests and multiplicative homomorphism can be connected how follows:

Lemma 31. *The surmise relation between tests $\dot{\mathscr{S}}$ on the set $\mathscr{T}$ of tests is transitive whenever the image matrix is a multiplicative homomorphism.*

Proof.

Consider the Boolean matrix $B{:=}MM$.

$\forall i,k \in \{1,\dots,n\}$: $B_{ik} = \sum_{j}^{n} M_{ij}M_{jk}$.

Suppose $B_{ik} = 1 \Leftrightarrow \exists j \in \{1,\dots,n\}$: $M_{ij} = M_{jk} = 1$.

S is transitive $\Rightarrow (M_{ij} = 1 \wedge M_{jk} = 1 \Rightarrow M_{ik} = 1 \quad \forall i,j,k \in \{1,\dots,n\})$.

Thus, $B_{ik} = 1 \Rightarrow M_{ik} = 1 \quad \forall i,k \in \{1,\dots,n\}$.

Suppose $B_{ik} = 0 \Rightarrow (\forall j \in \{1,\dots,n\}$: $M_{ij} = 1 \Rightarrow M_{jk} = 0)$.

Assume $M_{ik} = 1 \Rightarrow M_{kk} = 0$.

But S is reflexive, and thus, $M_{jj} = 1 \quad \forall j \in \{1,\dots,n\}$.

Thus, $B_{ik} = 0 \Rightarrow M_{ik} = 0 \quad \forall i,k \in \{1,\dots,n\}$

$\Rightarrow B = M \Rightarrow MM = M$.

The image matrix is a multiplicative homomorphism

$\Rightarrow \dot{M}\dot{M} = \dot{M} \Rightarrow \forall A,C \in \mathscr{T}$: $\dot{M}_{AC} = \sum_{B\in\mathscr{T}} \dot{M}_{AB}\dot{M}_{BC}$

$\Rightarrow (\forall A,B,C \in \mathscr{T}$: $\dot{M}_{AB} = 1 \wedge \dot{M}_{BC} = 1 \Rightarrow \dot{M}_{AC} = 1)$

$\Rightarrow \dot{\mathscr{S}}$ is transitive. □

The special conditions G_1 and G_q (where q is the cardinality of Q) are equivalent to the right- and left-covering conditions, respectively.

Lemma 32. *The surmise relation $\dot{\mathscr{S}}$ on the set $\mathscr{T}$ of tests satisfies $G_1 \Leftrightarrow \dot{\mathscr{S}}$ is right-covering.*

Proof.

Let $A,B \in \mathscr{T}$, $A\,\dot{\mathscr{S}}\,B$.

$\dot{\mathscr{S}}$ satisfies $G_1 \Leftrightarrow \forall X \subseteq A \quad$ with $|X| = 1$: $|\{b \in B \mid \exists a \in X$: $M_{ab} = 1\}| \geqslant 1$

$\Leftrightarrow \forall X = \{a\} \subseteq A$: $|\{b \in B \mid M_{ab} = 1\}| \geqslant 1$

$\Leftrightarrow \forall a \in A\ \exists b \in B$: $M_{ab} = 1 \Leftrightarrow A\,\dot{\mathscr{S}}_r\,B$ (see Proof 5). □

Lemma 33. *Let $|Q| = q$. The surmise relation $\dot{\mathscr{S}}$ on the set $\mathscr{T}$ of tests satisfies $G_q \Leftrightarrow \dot{\mathscr{S}}$ is left-covering.*

Proof.

$$\begin{aligned}
&\text{Let } A,B\in\mathscr{T},\ A\,\dot{\mathscr{S}}\,B.\\
&\dot{\mathscr{S}} \text{ satisfies } G_q \Leftrightarrow \forall X = A\colon |\{b\in B \mid \exists a\in X\colon M_{ab}=1\}| \geqslant |B|\\
&\quad\Leftrightarrow \{b\in B\ \exists a\in A\colon M_{ab}=1\} = B\\
&\quad\Leftrightarrow \forall b\in B\ \exists a\in A\colon M_{ab}=1\\
&\quad\Leftrightarrow A\,\dot{\mathscr{S}}_l\,B \text{ (see Proof 5).} \qquad \square
\end{aligned}$$

We can also use Boolean matrices for the representation of knowledge structures. A knowledge structure $\mathscr{K} = \{K_1,\ldots,K_n\}$ on the set $Q = \{x_1,\ldots,x_m\}$ can be represented by an $n\times m$ Boolean matrix X, whose entries are defined by $X_{ij}=1$ if knowledge state K_i contains the item $x_j\in Q$, and $X_{ij}=0$, otherwise, for $i=1,\ldots,n$ and $j=1,\ldots,m$. Partitioning the columns of X into tests establishes the relation between the knowledge structure and the corresponding test knowledge structure.

Using the matrix representation by Kim and Roush in addition to the set- and relation-oriented notation well-established in knowledge space theory, the range of applications of surmise relations between tests is enlarged, computations may be realizable in more efficient procedures, and proofs may become more elegant. However, we cannot do without the set- and relation-oriented notation as it is the most usual in this field.

6. Further research and possible interpretations

As the previous section shows, the reformulation of knowledge space theory by matrices is an important issue for further research in this field with respect to facilitating further mathematical developments as well as to the implementation of efficient software procedures.

In addition, by means of the results presented in this paper we want to find efficient ways for partitioning sets of items into tests regarding mathematical criteria as antisymmetry, transitivity and left- and right-coveringness as well as content-oriented criteria. Furthermore, we want to investigate interdependencies and parallelity for tests.

Furthermore, we want to generalize the concept of surmise relations between tests to surmise *systems* between tests, which allow different ways of solving a problem. Besides that, we want to establish principles for handling data—especially noisy data. In general, empirically obtained data are noisy, e.g. because of careless errors and lucky guesses or because of missing data. Methods for handling such data must be found. This mathematical model will be a basis for a software system that will analyze tests as well as partition sets of items into tests. Finally the software system will be tested empirically by applying it to a set of standard intelligence tests.

The applicability of surmise relations between tests is not restricted to psychological tests. Besides the relationships between courses and curricula which were already mentioned, interpretations and applications may be in structuring e.g. hyper-texts, the organization of companies, or upward drawings.

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